Malaysian Journal of Mathematical Sciences 16(1): 1–10 (2022)
https://doi.org/10.47836/mjms.16.1.01

Malaysian Journal of Mathematical Sciences

Journal homepage: https://einspem.upm.edu.my/journal

The Difference Splitting Scheme for n-Dimensional Hyperbolic Systems

Aloev R. D. *[1], Eshkuvatov Z. K.[2], Khudoyberganov M. U.[1], and Nematova D. E.[1]

[1]*Faculty of Mathematics, National University of Uzbekistan, Uzbekistan*
[2]*Faculty of Science and Technology, Universiti Sains Islam Malaysia, Malaysia*

E-mail: aloevr@mail.ru
**Corresponding author*

Received: 10 March 2020
Accepted: 14 October 2021

Abstract

In this paper, we propose the difference splitting scheme for a mixed problem posed for n -dimensional symmetric t-hyperbolic systems. We construct the difference splitting scheme for the numerical calculation of stable solutions for this system. To build a difference scheme, a multidimensional problem is split into one-dimensional ones and solved for each direction. A discrete analogue of the Lyapunov's function is constructed for the numerical verification of stability solutions for the considered problem. A priori estimate is obtained for the discrete analogue of the Lyapunov's function. This estimate allows us to assert the exponential stability of the numerical solution. A theorem on the exponential stability solution of the boundary value problem for linear hyperbolic system was proved. These stability theorems give us the opportunity to prove the convergence of the numerical solution.

Keywords: multidimensional hyperbolic systems; stability; difference scheme; splitting scheme.

1 Introduction

We consider the mixed dissipative boundary value problem for a n-dimensional linear hyperbolic system with variable coefficients [3]. For this problem, we construct and investigate the difference splitting scheme in order to obtain stable solutions. A discrete analogue of the Lyapunov function is constructed and a priori estimate is obtained for difference splitting scheme. The obtained a-priori estimate allows us to assert the exponential stability of the numerical solution. It should be noted that numerous problems have been devoted to the solution of such problems (see [1] and references there in). The stability solutions for one-dimensional hyperbolic systems were studied in the subject of Bastin and Coron ([2]). In the next section, it will be described the exponential stability of multidimensional hyperbolic systems. The difference scheme will be constructed and the stability of this scheme will be proved to find a numerical solution to the differential problem.

2 Differential Statement of the Problem

In the domain $G = \{(t, x_1, x_2, ...x_n) : 0 < t \leq T, 0 < x_i < X_i, i = 1, ...n\}$, we consider a symmetric hyperbolic system in a special canonical form as follows:

$$\frac{\partial \mathbf{v}}{\partial t} + \sum_{i=1}^{n} \mathbf{B}_i \frac{\partial \mathbf{v}}{\partial x_i} + \mathbf{Q}\mathbf{v} = 0, \tag{1}$$

with boundary conditions at $x_1 = 0:$

$$\mathbf{v}^{\mathbf{I}}(t, 0, x_2, \cdots, x_n) = \mathbf{s}\mathbf{v}^{\mathbf{II}}(t, 0, x_2, \cdots, x_n), \tag{2}$$
$$0 < t \leq T, \;\; 0 \leq x_i \leq X_i, \;\; i = 2, \cdots, n;$$

and at $x_1 = X_1:$

$$\mathbf{v}^{\mathrm{II}}(t, X_1, x_2, \cdots, x_n) = \mathbf{r}\mathbf{v}^{\mathrm{I}}(t, X_1, x_2, \cdots, x_n), \tag{3}$$
$$0 < t \leq T, \;\; 0 \leq x_i \leq X_i, i = 2, \cdots, n;$$

and periodicity condition at $x_i = 0, \; X_i\,; \; i = 2, \cdots, n :$ is

$$\mathbf{v}(t, \mathbf{x})|_{x_i=0} = \mathbf{v}(t, \mathbf{x})|_{x_i=X_i}, \; i = 2, 3, \cdots, n, \tag{4}$$

for $0 < t \leqslant T, \;\; 0 \leqslant x_I \leqslant X_I, \;\; I = 1, 2, 3, \cdots, n; \;\; (I \neq i)$ with initial data at $t = 0$ given by

$$\mathbf{v}(0, \mathbf{x}) = \phi(\mathbf{x}), \quad \mathbf{x} = (x_1, \cdots x_n), \quad 0 \leq x_i \leq X_i, \quad i = 1, \cdots, n, \tag{5}$$

where $\mathbf{v}^{\mathrm{I}} = (v_1, v_2, \cdots, v_m)^{\mathrm{T}}$ and $\mathbf{v}^{\mathrm{II}} = (v_{m+1}, v_{m+2}, \cdots, v_q)^{\mathrm{T}}$, are vectors to be determined; $\mathbf{B_1}(x_1)$ is a diagonal matrix; $\mathbf{B}_i(x_1) = \mathbf{B}_i^*(x_1), \;\; i = 2, \cdots, n$ are symmetric matrices; $\mathbf{Q}(\mathbf{x_1})$ is a square matrix of dimension $q \times q$, as well as $\phi(\mathbf{x}) = (\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x}), \cdots, \varphi_q(\mathbf{x}))^T$ - are given vector function. Let the entities be given in the form

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}^{\mathrm{I}} \\ \mathbf{v}^{\mathrm{II}} \end{bmatrix}, \qquad \mathbf{B}_1(x_1) = \begin{pmatrix} \mathbf{B}_1^{+}(x_1) & 0 \\ 0 & -\mathbf{B}_1^{-}(x_1) \end{pmatrix},$$

where

$$\mathbf{B}_1^+(x_1) = diag\left(b_1(x_1), b_2(x_1), ..., b_m(x_1)\right),$$
$$\mathbf{B}_1^-(x_1) = diag\left(b_{m+1}(x_1), b_{m+2}(x_1), ..., b_q(x_1)\right),$$

with

$$b_1(x_1) \geq ... \geq b_m(x_1) > 0 \geq -b_{m+1}(x_1) \geq ... \geq -b_q(x_1),$$

and $\mathbf{s}$ is a matrix of the size $(q-m)\times m$, $\mathbf{r}$ is a matrix of order $m \times (q-m)$.

Definition 2.1. (*Exponential stability*). *The system* (1) *with boundary conditions* (2) - (4) *is exponentially stable in the* L^2 *-norm if there exists such* $\nu > 0$ *and* $c > 0$ *that for any initial condition* $\phi \in L^2$ $(0 \leqslant x_i \leqslant X_i, \quad i=1,\cdots,n;\ R^q)$, *the* L^2 *-solution of the mixed problem* (1) - (5) *satisfies the inequality*

$$\|\mathbf{v}(t,\cdot)\|_{L^2(0\leqslant x_i \leqslant X_i,\ \ i=1,\cdots,n;R^q)} \leqslant ce^{-\nu t}\,\|\phi\|_{L^2(0\leqslant x_i\leqslant X_i,\, i=1,\cdots,n;R^q)} \quad \textit{for } t\geqslant 0. \tag{6}$$

We construct the Lyapunov's function in the form:

$$\begin{aligned} L(t) &= \int_0^{\mathbf{X}} \left(\mu(x_1)\mathbf{v},\mathbf{v}\right)\mathbf{dx} \\ &= \int_0^{X_n}\cdots\int_0^{X_1} \left(\mu(x_1)\mathbf{v},\mathbf{v}\right)\, dx_1\cdots dx_n \\ &= \int_0^{X_n}\cdots\int_0^{X_1} \left\{\sum_{i=1}^{m}\mu_i e^{-\nu x_1}\left[v_i(t,\mathbf{x})\right]^2 + \sum_{i=m+1}^{q}\mu_i e^{\nu x_1}\left[v_i(t,\mathbf{x})\right]^2\right\} dx_1\cdots dx_n, \end{aligned} \tag{7}$$

where

$$\begin{aligned} \mu_i &> 0, \\ i &= 1,\dots,q, \\ \mu^+ &= diag(\mu_1,\dots,\mu_m), \\ \mu^- &= diag(\mu_{m+1},\dots,\mu_q), \\ \mu(x_1) &= \begin{pmatrix} e^{-\nu x_1}\mu^+ & 0 \\ 0 & e^{\nu x_1}\mu^- \end{pmatrix}. \end{aligned}$$

Theorem 2.1. (*Exponential stability*). *The system* (1) *with the boundary conditions* (2) - (4) *is exponentially stable in the* L^2 *-norm if there exists* $\nu > 0$ *and* $\mu_i > 0,\ i=1,\dots,q$ *such that the following matrices*

$$\begin{pmatrix} \mathbf{B}_1^+(X_1) & 0 \\ 0 & \mathbf{B}_1^-(0) \end{pmatrix}\begin{pmatrix} e^{-\nu X_1}\mu^+ & 0 \\ 0 & \mu^- \end{pmatrix} -$$

$$\begin{pmatrix} 0 & \mathbf{r} \\ \mathbf{s} & 0 \end{pmatrix}\begin{pmatrix} \mathbf{B}_1^+(0) & 0 \\ 0 & \mathbf{B}_1^-(X_1) \end{pmatrix}\begin{pmatrix} \mu^+ & 0 \\ 0 & e^{\nu X_1}\mu^- \end{pmatrix}\begin{pmatrix} 0 & \mathbf{s} \\ \mathbf{r} & 0 \end{pmatrix}, \tag{8}$$

and

$$\nu\left|\mathbf{B}_1(x_1)\right|\mu(x_1) - \mathbf{B}_1^{'}(x_1)\mu(x_1), \quad x_1 \in (0, X_1), \tag{9}$$

are positive definite, as well as form $(\mu(x_1)\mathbf{Qv},\mathbf{v}) > 0$.

Proof. We calculate the derivative of the Lyapunov function:

$$\begin{aligned} L'(t) &= \int_0^{X_n} \cdots \int_0^{X_1} \partial_t \left(\mu(x_1)\mathbf{v}, \mathbf{v}\right) \, dx_1 \cdots dx_n \\ &= -\int_0^{X_n} \cdots \int_0^{X_1} \left[(\mu(x_1)\mathbf{B}_1(x_1)\partial_{x_1}\mathbf{v}, \mathbf{v}) + (\mu(x_1)\mathbf{v}, \mathbf{B}_1(x_1)\partial_{x_1}\mathbf{v})\right] dx_1 \cdots dx_n \\ &\quad + \int_0^{X_n} \cdots \int_0^{X_1} \sum_{i=2}^{n} \{(\mu(x_1)\mathbf{B}_i(x_1)\partial_{x_i}\mathbf{v}, \mathbf{v}) + (\mu(x_1)\mathbf{v}, \mathbf{B}_i(x_1)\partial_{x_i}\mathbf{v})\} dx_1 \cdots dx_n \\ &\quad - \int_0^{X_n} \cdots \int_0^{X_1} \left[(\mu(x_1)\mathbf{Q}\mathbf{v}, \mathbf{v}) + (\mu(x_1)\mathbf{v}, \mathbf{Q}\mathbf{v})\right] dx_1 \cdots dx_n. \end{aligned}$$

Here we used the equations (1), since

$$(\mu(x_1)\mathbf{B}_1(x_1)\partial_{x_1}\mathbf{v}, \mathbf{v}) = \left(\partial_x \left[\mu(x_1)\mathbf{B}_1(x_1)\mathbf{v}\right], \mathbf{v}\right) - (\mu'(x_1)\mathbf{B}_1(x_1)\mathbf{v}, \mathbf{v}) - \left(\mu(x_1)\mathbf{B}_1'(x_1)\mathbf{v}, \mathbf{v}\right),$$

and

$$\mu'(x_1)\mathbf{B}_1(x_1) = -\nu \left|\mathbf{B}_1(x_1)\right| \mu(x_1),$$

we have the following identity:

$$\begin{aligned} L'(t) = &- \int_0^{X_n} \cdots \int_0^{X_2} \left(\mathbf{B}_1(x_1)\mu(x_1)\mathbf{v}, \mathbf{v}\right)\Big|_0^{X_1} dx_2 \cdots dx_n \\ &- \int_0^{X_n} \cdots \int_0^{X_1} \left(\left[\nu \left|\mathbf{B}_1(x_1)\right| \mu(x_1) - \mathbf{B}_1'(x_1)\mu(x_1)\right] \mathbf{v}, \mathbf{v}\right) \, dx_1 \cdots dx_n \\ &- 2\int_0^{X_n} \cdots \int_0^{X_1} \left(\mu(x_1)\mathbf{Q}\mathbf{v}, \mathbf{v}\right) dx_1 \cdots dx_n. \end{aligned}$$

Using the periodicity of the boundary conditions, we obtained

$$\int_0^{X_n} \cdots \int_0^{X_3} \int_0^{X_1} \left(\mathbf{B}_2(x_1)\mu(x_1)\mathbf{v}, \mathbf{v}\right)\Big|_0^{X_2} dx_1 dx_3 \cdots dx_n = 0,$$

$$\vdots$$

$$\int_0^{X_{n-1}} \cdots \int_0^{X_1} \left(\mathbf{B}_n(x_1)\mu(x_1)\mathbf{v}, \mathbf{v}\right)\Big|_0^{X_n} dx_1 \cdots dx_{n-1} = 0.$$

Now, transform separately each term of the penultimate then we obtained identity:

$$-\left(\mathbf{B}_1(x_1)\mu(x_1)\mathbf{v}, \mathbf{v}\right)\Big|_0^{X_1} < 0.$$

Here $\mathbf{y} = (x_2, \cdots, x_n)^T$ according to (9), we have

$$-\int_0^{X_n} \cdots \int_0^{X_1} \left(\left[\nu \left|\mathbf{B}_1(x_1)\right| \mu(x_1) - \mathbf{B}_1'(x_1)\mu(x_1)\right] \mathbf{v}, \mathbf{v}\right) \, dx_1 \cdots dx_n -$$

$$2\int_0^{X_n} \cdots \int_0^{X_1} \left(\mu(x_1)\mathbf{Q}\mathbf{v}, \mathbf{v}\right) dx_1 \cdots dx_n < 0.$$

Taking into account these transformations, we obtain

$$\begin{aligned} L'(t) = &- \int_0^{X_n} \cdots \int_0^{X_2} \left(\mathbf{B}_1(x_1)\mu(x_1)\mathbf{v},\mathbf{v}\right)\Big|_0^{X_1} dx_2\cdots dx_n \\ &- \int_0^{X_n} \cdots \int_0^{X_1} \left(\left[\nu\left|\mathbf{B}_1(x_1)\right|\mu(x_1) - \mathbf{B}_1'(x_1)\mu(x_1)\right]\mathbf{v},\mathbf{v}\right)\, dx_1\cdots dx_n \\ &- 2\int_0^{X_n} \cdots \int_0^{X_1} \left(\mu(x_1)\mathbf{Q}\mathbf{v},\mathbf{v}\right) dx_1\cdots dx_n < 0. \end{aligned}$$

Since the matrices (9) and $\nu\left|\mathbf{B}_1(x_1)\right|\mu(x_1)$ are positive definite for any x_1, we get following inequality

$$\begin{gathered} \int_0^{X_n} \cdots \int_0^{X_1} \left(\left[\nu\left|\mathbf{B}_1(x_1)\right|\mu(x_1) - \mathbf{B}_1'(x_1)\mu(x_1)\right]\mathbf{v},\mathbf{v}\right)\, dx_1\cdots dx_n + \\ 2\int_0^{X_n} \cdots \int_0^{X_1} \left(\mu(x_1)\mathbf{Q}\mathbf{v},\mathbf{v}\right) dx_1\cdots dx_n > \\ \int_0^{X_n} \cdots \int_0^{X_1} \left(\nu\left|\mathbf{B}_1(x_1)\right|\mu(x_1)\mathbf{v},\mathbf{v}\right)\, dx_1\cdots dx_n > \\ b\nu\int_0^{X_n} \cdots \int_0^{X_1} \left(\mu(x_1)\mathbf{v},\mathbf{v}\right)\, dx_1\cdots dx_n \end{gathered}$$

where $b = \min\limits_{\substack{1\leqslant i\leqslant n\\ 0\leqslant x_1\leqslant X_1}} |b_i(x_1)|$ and for any vector $\mathbf{v}\in R^q$ we have

$$L'(t) < -\nu b\int_0^{X_n} \cdots \int_0^{X_1} \left(\mu(x_1)\mathbf{v},\mathbf{v}\right)\, dx_1\cdots dx_n = -\eta L(t), \quad \eta = \nu b.$$

Hence,

$$L(t) \leqslant e^{-\eta t}L(0), \quad t > 0.$$

However, there is such a constant $\gamma > 0$ such that

$$\frac{1}{\gamma}\left\|\mathbf{v}(t,\cdot)\right\|^2_{L^2(0\leqslant x_i\leqslant X_i,\ \ i=1,\cdots,n\,;\,R^q)} \leqslant L(t) \leqslant \gamma\left\|\mathbf{v}(t,\cdot)\right\|^2_{L^2(0\leqslant x_i\leqslant X_i,\ \ i=1,\cdots,n\,;\,R^q)},$$

$$\left\|\mathbf{v}(t,\cdot)\right\|^2_{L^2(0\leqslant x_i\leqslant X_i,\ \ i=1,\cdots,n\,;\,R^q)} = \int_0^{X_n} \cdots \int_0^{X_1} \left(\mathbf{v},\mathbf{v}\right) dx_1\cdots dx_n,$$

then we obtain

$$\left\|\mathbf{v}(t,\cdot)\right\|_{L^2(0\leqslant x_i\leqslant X_i,\ \ i=1,\cdots,n\,;\,R^q)} \leqslant \gamma e^{-\eta t/2}\left\|\phi\right\|_{L^2(0\leqslant x_i\leqslant X_i,\ \ i=1,\cdots,n\,;\,R^q)}, \quad t\in[0,+\infty).$$

□

3 The Difference Scheme

In the domain G, construct the difference grid

$$G_h = \{(t^\kappa, x_i^j): \ \ 0 \leqslant t^\kappa \leqslant T, \ \ 0 \leqslant x_i^j \leqslant X_i\}:$$

where

$$t^\kappa = \kappa\,\Delta\, t, \quad \kappa = 0,\dots,N_t; \quad N_t\,\Delta\, t = T, x_i^j = (j_i + \frac{1}{2})\Delta x_i,$$
$$J_i \Delta x_i = X_i; \ \ j_i = 0,\dots,J_i - 1; \ \ i = 1,\cdots,n;$$

and denote the value of the numerical solution at the nodal points by

$$\mathbf{v}_{\mathbf{j}}^\kappa = \prod_{i=1}^{n} \frac{1}{\Delta x_i} \int_{x_n^{j_n - \frac{1}{2}}}^{x_n^{j_n + \frac{1}{2}}} \cdots \int_{x_1^{j_1 - \frac{1}{2}}}^{x_1^{j_1 + \frac{1}{2}}} \mathbf{v}(t^\kappa, \mathbf{x}) d\mathbf{x},$$

where

$$j_i = 0,\dots,J_i - 1; \ \ i = \ 1,\dots,n; \ \mathbf{x} = (x_1,\cdots,x_n)\,; \ d\mathbf{x} = dx_1\cdots dx_n; \ \ \mathbf{j} = j_1\cdots j_n\,.$$

For the numerical solution of the mixed problem (1) - (5), we suggest the difference splitting scheme

$$\begin{bmatrix} \left(\mathbf{v}_1^{\mathbf{I}}\right)_{\mathbf{j}}^{\kappa} \\ \left(\mathbf{v}_1^{\mathbf{II}}\right)_{\mathbf{j}}^{\kappa} \end{bmatrix} = \begin{bmatrix} \left(\mathbf{v}^{\mathbf{I}}\right)_{\mathbf{j}}^{\kappa} \\ \left(\mathbf{v}^{\mathbf{II}}\right)_{\mathbf{j}}^{\kappa} \end{bmatrix} - \frac{\Delta t}{\Delta x_1} \begin{bmatrix} \left(\mathbf{B}_1^{+}\right)_{j_1 - 1} & 0 \\ 0 & \left(\mathbf{B}_1^{-}\right)_{j_1 + 1} \end{bmatrix} \begin{bmatrix} \left(\mathbf{v}^{\mathbf{I}}\right)_{\mathbf{j}}^{\kappa} - \left(\mathbf{v}^{\mathbf{I}}\right)_{j_1 - 1}^{\kappa} \\ \left(\mathbf{v}^{\mathbf{II}}\right)_{\mathbf{j}}^{\kappa} - \left(\mathbf{v}^{\mathbf{II}}\right)_{j_1 + 1}^{\kappa} \end{bmatrix}, \tag{10}$$

$$\left(\mathbf{v}_{i+1}\right)_{\mathbf{j}}^{\kappa} = \left(\mathbf{v}_i\right)_{\mathbf{j}}^{\kappa} - \frac{\Delta t}{\Delta x_{i+1}} \begin{bmatrix} \left(\mathbf{B}_{i+1}^{+}\right)_{j_1} & 0 \\ 0 & \left(\mathbf{B}_{i+1}^{-}\right)_{j_1} \end{bmatrix} \begin{bmatrix} \left(\mathbf{v}_i^{\mathbf{I}}\right)_{\mathbf{j}}^{\kappa} - \left(\left(\mathbf{v}_i^{\mathbf{I}}\right)\right)_{j_i - 1}^{\kappa} \\ \left(\mathbf{v}_i^{\mathbf{II}}\right)_{\mathbf{j}}^{\kappa} - \left(\mathbf{v}_i^{\mathbf{II}}\right)_{j_i + 1}^{\kappa} \end{bmatrix}; \ \ i = 1,\cdots,n-1;$$

where

$$\mathbf{v}_{\mathbf{j}}^{\kappa+1} = \left(\mathbf{v}_n\right)_{\mathbf{j}}^{\kappa} - \Delta t \mathbf{Q} \left(\mathbf{v}_n\right)_{\mathbf{j}}^{\kappa}. \tag{11}$$

$$\left(\mathbf{v}^{\mathbf{I}}\right)_{j_1 \pm 1}^{\kappa} = \left(\mathbf{v}^{\mathbf{I}}\right)_{j_1 \pm 1, j_2, \cdots, j_n}^{\kappa}; \quad \left(\mathbf{v}^{\mathbf{II}}\right)_{j_1 \pm 1}^{\kappa} = \left(\mathbf{v}^{\mathbf{II}}\right)_{j_1 \pm 1, j_2, \cdots, j_n}^{\kappa};$$
$$\left(\mathbf{B}_1^{\pm}\right)_{j_1 \mp 1} = \left(\mathbf{B}_1^{\pm}\right)_{j_1 \mp 1, j_2, \cdots, j_n}; \left(\mathbf{v}_i\right)_{j_i \pm 1}^{\kappa} = \mathbf{v}_{j_1 \cdots j_{i-1} j_i \pm 1, j_{i+1} \cdots j_n}^{\kappa};$$
$$\mathbf{B}_i = \begin{pmatrix} \mathbf{B}_i^{+} & 0 \\ 0 & -\mathbf{B}_i^{-} \end{pmatrix}, \ \mathbf{B}_i^{\pm} \geqslant 0; \ \ j_i = 0,\dots,J_i - 1; \ \ \kappa = 0,\dots,N-1.$$

The initial conditions (5) are approximated as follows:

$$\mathbf{v}_{\mathbf{j}}^{0} = \prod_{i=1}^{n} \frac{1}{\Delta x_i} \int_{x_n^{j_n - \frac{1}{2}}}^{x_n^{j_n + \frac{1}{2}}} \cdots \int_{x_1^{j_1 - \frac{1}{2}}}^{x_1^{j_1 + \frac{1}{2}}} \phi(\mathbf{x}) d\mathbf{x}, j_i = 0,\dots,J_i - 1, i = \ 1,\dots,n, \tag{12}$$

and the boundary conditions are approximated in this way:

$$\begin{pmatrix} (\mathbf{v}^{\mathbf{I}})^{\kappa+1}_{-1,\mathbf{j}} \\ (\mathbf{v}^{\mathbf{II}})^{\kappa+1}_{J_1,\mathbf{j}} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{s} \\ \mathbf{r} & 0 \end{pmatrix} \begin{pmatrix} (\mathbf{v}^{\mathbf{I}})^{\kappa+1}_{J_1-1,\mathbf{j}} \\ (\mathbf{v}^{\mathbf{II}})^{\kappa+1}_{0,\mathbf{j}} \end{pmatrix},$$

$$(\mathbf{v}^{\mathbf{I}}_i)^{\kappa}_0 = (\mathbf{v}^{\mathbf{I}}_i)^{\kappa}_{J_i-1}, \quad (\mathbf{v}^{\mathbf{II}}_i)^{\kappa}_1 = (\mathbf{v}^{\mathbf{II}}_i)^{\kappa}_{J_i}; \; \kappa = 0, \cdots, N-1; \; \mathbf{j} = j_2 \cdots j_n; \; i = 1, \cdots, n. \tag{13}$$

Suppose that the condition of the CFL $\frac{\Delta t}{\Delta x_1} \max\limits_{\substack{1 \leqslant k \leqslant q \\ 0 \leqslant j_1 \leqslant J_1-1}} \left|(b_k)_{j_1}\right| \leqslant 1, \frac{\Delta t}{\Delta x_i} \max\limits_{\substack{2 \leqslant i \leqslant n \\ 1 \leqslant k \leqslant q \\ 0 \leqslant j_1 \leqslant J_1-1}} \left|\lambda_k(B_i^{\pm})_{j_1}\right| \leqslant 1$ holds. Here $(b_k)_{j_1} = b_k(x_1^{j_1})$, $(B_i^{\pm})_{j_1} = B_i^{\pm}\left(x_1^{j_1}\right)$ and $\lambda_k(B_i^{\pm})_{j_1}$ - are the eigenvalues of the matrix $\left(B_i^{\pm}\right)_{j_1}$. Now, we investigate the exponential stability for the solution of the difference problem (10) - (13).

Definition 3.1. *The difference scheme (10) - (12) with the difference boundary condition (13) is exponentially stable if there exist constants $\eta > 0$ and $c > 0$ such that for any initial condition*

$$\mathbf{v}^0_{\mathbf{j}} \in L^2\left(\left\{x_i^{j_i}\right\}, i = 1, \cdots, n; j_i = 0, \cdots J_i - 1; R^q\right),$$

the solution of the difference boundary value problem (10) - (13) satisfies the equality

$$\prod_{i=1}^{n} \Delta x_n \cdot \prod_{i=1}^{n} \sum_{j_i=1}^{J_i-1} (\mathbf{v}^{\kappa}_{\mathbf{j}}, \mathbf{v}^{\kappa}_{\mathbf{j}}) \leqslant e^{-\eta t_\kappa} \prod_{i=1}^{n} \Delta x_n \cdot \prod_{i=1}^{n} \sum_{j_i=1}^{J_i-1} (\mathbf{v}^{0}_{\mathbf{j}}, \mathbf{v}^{0}_{\mathbf{j}}), \quad \kappa = 1, \ldots, N.$$

Consider the difference boundary-value problem (10) - (13) with the stationary solution

$$\mathbf{v}^{\kappa}_{\mathbf{j}} = 0, \quad \kappa = 0, \ldots, N-1; \quad j_i = 0, \ldots, J_i - 1; \quad i = 1, \ldots, n.$$

In order to prove the stability of the difference boundary-value problem (11) - (13), we propose the following function as the discrete Lyapunov function

$$L(\mathbf{v}^{\kappa}) = L^{\kappa} = \prod_{i=1}^{n} \Delta x_i \cdot \prod_{i=1}^{n} \sum_{j_i=1}^{J_i-1} (\mathbf{v}^{\kappa}_{\mathbf{j}}, \mu_{j_1} \mathbf{v}^{\kappa}_{\mathbf{j}}), \tag{14}$$

where

$$\mu_{j_1} = \mu(x_{j_1}),$$

$$j_1 = 1, \cdots, J_1 - 1,$$

$$\prod_{i=1}^{n} \Delta x_i = \Delta x_1 \cdots \Delta x_n,$$

$$\prod_{i=1}^{n} \sum_{j_i=1}^{J_i-1} (\mathbf{v}^{\kappa}_{\mathbf{j}}, \mu_{j_1} \mathbf{v}^{\kappa}_{\mathbf{j}}) = \sum_{j_1=1}^{J_1-1} \cdots \sum_{j_n=1}^{J_n-1} (\mathbf{v}^{\kappa}_{\mathbf{j}}, \mu_{j_1} \mathbf{v}^{\kappa}_{\mathbf{j}}),$$

$$\mu_{j_1} = \begin{pmatrix} e^{-\nu x_1^{j_1}} \mu^{+} & 0 \\ 0 & e^{\nu x_1^{j_1}} \mu^{-} \end{pmatrix}.$$

Theorem 3.1. *Let $T > 0$ and the discrete Lyapunov function is defined by (14). If the condition of CFL*

$$\frac{\Delta t}{\Delta x_1} \max_{\substack{1\leqslant k\leqslant q \\ 0\leqslant j_1\leqslant J_1-1}} \left|(b_k)_{j_1}\right| \leqslant 1, \frac{\Delta t}{\Delta x_1} \max_{\substack{2\leqslant i\leqslant n \\ 1\leqslant k\leqslant q \\ 0\leqslant j_1\leqslant J_1-1}} \left|\lambda_k(B_i^{\pm})_{j_1}\right| \leqslant 1,$$

holds, and there exist real numbers $\nu > 0$ and $\mu_i > 0,\ i = 1, \ldots, n$, such that $0 < \eta \equiv \nu\alpha e^{-\nu\Delta x_1} - \beta < 1$, where

$$\alpha = \frac{\Delta t}{\Delta x_1} \max_{\substack{1\leqslant k\leqslant q \\ 0\leqslant j_1\leqslant J_1-1}} \left|(b_k)_{j_1}\right|,$$

$$\beta = \max_{\substack{1\leqslant k\leqslant q \\ 0\leqslant j_1\leqslant J_1-1}} \left|(b_k')_{j_1}\right|,$$

$$2(\mathbf{Q}_{j_1}\mathbf{u}_{\mathbf{j}}^{\kappa}, \mu_{j_1}\mathbf{u}_{\mathbf{j}}^{\kappa}) - \Delta t(\mathbf{Q}_{j_1}\mathbf{u}_{\mathbf{j}}^{\kappa}, \mu_j\mathbf{Q}_{j_1}\mathbf{u}_{\mathbf{j}}^{\kappa}),\ j_1 = 0, \ldots, J_1 - 1,$$

are non-negative, and

$$\begin{bmatrix} \mu^+ e^{-\nu x_1^{J_1}} (\mathbf{B}_1)^+_{J_1-1} & 0 \\ 0 & \mu^- e^{\nu x_1^{-1}} (\mathbf{B}_1)^-_0 \end{bmatrix} - \begin{pmatrix} 0 & \mathbf{r} \\ \mathbf{s} & 0 \end{pmatrix} \begin{bmatrix} \mu^+ e^{-\nu x_1^0} (\mathbf{B}_1)^+_{-1} & 0 \\ 0 & \mu^- e^{\nu x_1^{J_1-1}} (\mathbf{B}_1)^-_{J_1} \end{bmatrix} \begin{pmatrix} 0 & \mathbf{s} \\ \mathbf{r} & 0 \end{pmatrix},$$

is a positive definite matrix, then the numerical solution $\mathbf{v}_{\mathbf{j}}^{\kappa}$ of the difference boundary value problem (11) - (14) converges to the stationary solution $\mathbf{v}_{\mathbf{j}}^{\kappa} = 0$ for the L^2 -norm.

Proof. Using the Lyapunov discrete function, we calculate the discrete derivative of the Lyapunov function (14) as follows

$$\frac{L(\mathbf{v}^{\kappa+1}) - L(\mathbf{v}^{\kappa})}{\Delta t} = \frac{L(\mathbf{v}^{\kappa+1}) - L(\mathbf{v}_n^{\kappa})}{\Delta t} + \frac{L(\mathbf{v}_n^{\kappa}) - L(\mathbf{v}_{n-1}^{\kappa})}{\Delta t} + \cdots + \frac{L(\mathbf{v}_1^{\kappa}) - L(\mathbf{v}^{\kappa})}{\Delta t},$$

where

$$L(\mathbf{v}^{\kappa}) = \prod_{i=1}^{n} \Delta x_i \cdot \prod_{i=1}^{n} \sum_{j_i=1}^{J_i-1} (\mathbf{v}_{\mathbf{j}}^{\kappa}, \mu_{j_1}\mathbf{v}_{\mathbf{j}}^{\kappa}),$$

$$L(\mathbf{v}_l^{\kappa}) = \prod_{i=1}^{n} \Delta x_i \cdot \prod_{i=1}^{n} \sum_{j_i=1}^{J_i-1} \left((\mathbf{v}_l)_{\mathbf{j}}^{\kappa}, \mu_{j_1}(\mathbf{v}_l)_{\mathbf{j}}^{\kappa}\right),\ l = 1, \cdots, n,\ \kappa = 0, \cdots, N.$$

Now we prove that this quadratic form is negatively defined. For this, it suffices to show that all quadratic forms on the right-hand side of the previous equality are negatively defined.

Lemma 3.1. *Let the conditions of Theorem 2 be satisfied. Then the quadratic form*

$$\frac{L(\mathbf{v}^{\kappa+1}) - L(\mathbf{v}_n^{\kappa})}{\Delta t} \leqslant 0$$

is not positively defined.

Lemma 3.2. *Let the conditions of Theorem 2 be satisfied. Then, the quadratic form*

$$\frac{L(\mathbf{v}_n^{\kappa}) - L(\mathbf{v}_{n-1}^{\kappa})}{\Delta t} \leqslant 0,$$

is not positively defined.

Lemma 3.3. *Let the conditions of Theorem 2 be satisfied. Then, the following inequality*

$$\frac{L(\mathbf{v}_1^\kappa) - L(\mathbf{v}^\kappa)}{\Delta t} < -\eta L(\mathbf{v}^\kappa),$$

holds.

As a result, Lemmas 3.1-3.3 imply the following inequalities:

$$\frac{L(\mathbf{v}^{\kappa+1}) - L(\mathbf{v}_n^\kappa)}{\Delta t} \leqslant 0, \quad \frac{L(\mathbf{v}_i^\kappa) - L(\mathbf{v}_{i-1}^\kappa)}{\Delta t} \leqslant 0, i = n-1, \cdots, 2; \quad \frac{L(\mathbf{v}_1^\kappa) - L(\mathbf{v}^\kappa)}{\Delta t} < -\eta L(\mathbf{v}^\kappa).$$

Summing up these inequalities, we have the following for the quadratic form

$$\frac{L^{\kappa+1} - L^\kappa}{\Delta t} < -\eta L^\kappa.$$

Recursively applying this inequality, we obtain

$$L^{\kappa+1} < (1 - \Delta t\eta)^{\kappa+1} L^0 \leqslant e^{-\eta\Delta t(\kappa+1)} L^0 = e^{-\eta t_{\kappa+1}} L^0, \ \kappa = 0, \ldots, N-1\,.$$

Denote

$$C_1 = \min_{\substack{1\leqslant i\leqslant q \\ 0\leqslant j_1\leqslant J_1-1}} \left\{\varpi_{ij_1} : |\mu_{j_1} - \varpi_{ij_1}\mathbf{E}| = 0\right\}, C_2 = \max_{\substack{1\leqslant i\leqslant q \\ 0\leqslant j_1\leqslant J_1-1}} \left\{\varpi_{ij_1} : |\mu_{j_1} - \varpi_{ij_1}\mathbf{E}| = 0\right\}.$$

Then,

$$C_1\mathbf{E} \leqslant \mu_{j_1} \leqslant C_2\mathbf{E}, \quad j_1 = 0, \ldots, J_1 - 1.$$

From this, it follows that

$$\prod_{i=1}^{n} \Delta x_i \cdot \prod_{i=1}^{n} \sum_{j_i=1}^{J_i-1} \left(\mathbf{v}_{\mathbf{j}}^\kappa, \mathbf{v}_{\mathbf{j}}^\kappa\right) \leqslant Ce^{-\eta t_\kappa} \prod_{i=1}^{n} \Delta x_i \cdot \prod_{i=1}^{n} \sum_{j_i=1}^{J_i-1} \left(\mathbf{v}_{\mathbf{j}}^0, \mathbf{v}_{\mathbf{j}}^0\right), \kappa = 1, \ldots, N; \quad C = C_2/C_1.$$

Thus, the numerical solution $\mathbf{v}_j^\kappa$ of the mixed problem is exponentially stable in the L_2 -norm. Theorem 2 is proved. □

4 Conclusions

In conclusion, we note that the stability of a difference splitting scheme was studied in the work for the numerical calculation of stable solutions of the mixed dissipative boundary value problem for n -dimensional linear hyperbolic system with variable coefficients. A discrete analogue of the Lyapunov function is constructed for the numerical value of stable solutions of the mixed dissipative boundary value problem for a n-dimensional linear hyperbolic system with variable coefficients. A priori estimate is obtained for the discrete analogue of the Lyapunov function. The obtained prior estimate allows us to assert the exponential stability of the numerical solution. Theorems on the exponential stability of the solution for both the differential problem and the difference splitting scheme for nonlinear hyperbolic systems in the corresponding norms are proved. Consequently, this gives us the opportunity to prove the convergence of a stable numerical solution to a stable solution of a differential problem.

Acknowledgements This work was supported by National University of Uzbekistan (NUUz) under Research Grant "Analysis of Li symmetry, analysis and modelling of the stability of hyperbolic systems on Lyapunov". Project code is UZB-Ind-2021-87.

Conflicts of Interest The authors declare no conflict of interest.

References

[1] R. D. Aloev, Z. K. Eshkuvatov, M. Khudoyberganov & D. E. Nematova (2019). The difference splitting scheme for hyperbolicsystems with variable coefficients. *Mathematics and Statistics*, *7*, 82–89.

[2] G. Bastin & J. M. Coron (2016). *Stability and boundary stabilization of 1-D hyperbolic systems*. Birkhäuser, Switzerland.

[3] S. K. Godunov (1979). *Equations of mathematical physics*. Nauka, Moscow.